On Thomae Function

Let I be any interval of finite positive length and let $n \in \mathbb{N}$. Let

$$Z_n := \{ m \in \mathbb{Z} : \tfrac{m}{n} \in I \}, \text{ where } \tfrac{m}{n} \text{ in lowest terms}$$

Then Z_n is finite (or empty) (by the well-order property, as Z_n is bounded). Hence

$$I_n := \{ x \in I \cap \mathbb{Q} : x = \tfrac{m}{n} \}$$ is also a finite set,

and so $\bigcup_{n \le N} I_n$ is finite for any $N \in \mathbb{N}$.

Let $f(0)=0$ and $f(x) = \begin{cases} 0 & \forall x \text{ irrational} \\ \frac{1}{n} & \forall x = \frac{m}{n} \text{ rational in lowest terms} \end{cases}$

Let $x_0 \in \mathbb{R} \setminus \mathbb{Q}$ (so $f(x_0) = 0$), and $\varepsilon > 0$. Let $I = (x_0 - \frac{1}{2}, x_0 + \frac{1}{2})$ and $N \in \mathbb{N}$ be such that $\frac{1}{N} < \varepsilon$. Since x_0 is not in the finite set $\bigcup_{n \le N} I_n$ (as x_0 is irrational and the set is contained in $\mathbb{Q}$), pick positive $\delta < \frac{1}{2}$ such that $V_\delta(x_0)$ disjoint from $\bigcup_{n \le N} I_n$

Then $|f(x) - f(x_0)| = f(x) < \varepsilon \quad \forall x \in V_\delta(x_0)$. This is because all rationals x in $V_\delta(x_0)$ are in $\bigcup_{n > N} I_n$ (so $f(x) = \frac{1}{n} < \frac{1}{N} < \varepsilon$).